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Power law tail in the radial growth probability distribution for DLA

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Abstract. Using both analytic and numerical methods, we study the radial growth probability distribution $P(r, M)$ for large-scale off-lattice diffusion-limited aggregation (DLA) clusters. If the form of $P(r, M)$ is a Gaussian, we show analytically that the width $\xi(M)$ of the distribution *cannot* scale as the radius of gyration R_G of the cluster. We generate about 1750 clusters of masses M up to 500 000 particles, and calculate the distribution by sending 10^6 further random walkers for each cluster. We give strong support that the calculated distribution has a power law tail in the interior ($r\sim 0$) of the cluster, and can be described by a scaling ansatz $P(r, M) \propto (r^\alpha/\xi) g((r-r_0)/\xi)$, where $g(x)$ denotes some scaling function which is centred around zero and has a width of order unity. The exponent α is determined to be ≈ 2, which is now substantially smaller than values measured earlier. We show, by including the power law tail, that the width *can* scale as R_G, if $\alpha > D_f - 1$.

1. Introduction

The growth of DLA [1] clusters can be described by a set of growth probabilities $\{p_i\}$. Each of them describes the probability that site i is touched by the next incoming particle. The determination of the p_i and their distribution has gained much interest [2–5]. Numerical calculations in this field were done by solving exactly the Laplace equation on a given DLA cluster [6, 7]. Unfortunately these calculations are limited by computer resources to cluster masses around 50 000 particles. An alternative quantity, which is more accessible by large-scale simulations, is the integrated radial growth probability $P(r, M)$. It describes the probability that the next incoming particle touches the cluster of mass M at a distance r from the seed. This quantity has been measured by Plischke and Racz [8] who studied the first two moments of the distribution and fitted a Gaussian behaviour

$$P(r, M) \propto \exp\left(-\frac{(r-r_0)^2}{2\xi^2}\right). \quad (1)$$

Using 4000 clusters of masses up to 2500 particles they obtain a power law behaviour of the centre r_0 and the width ξ of the Gaussian

$$r_0 \propto R_G \propto M^\nu \qquad \xi \propto M^{\nu'} \quad (2)$$

with exponents $\nu = 1/D_f \approx 0.585$, $\nu' \approx 0.48$, where D_f denotes the fractal dimension of the clusters and R_G is the radius of gyration. Later, using larger-scale simulations,

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Meakin and Sander [9] showed that the exponent ν' approaches ν with increasing cluster mass and hereby raised the question about the real behaviour of the width ξ. Thus, currently three different possibilities are being considered.

(1) Width scales with the same exponent ν as R_G: $\xi(M) \propto M^{\nu}$.
(2) Width scales with a smaller exponent ν': $\xi(M) \propto M^{\nu'}$, $\nu' < \nu$.
(3) Width scales with ν but having logarithmic corrections: $\xi(M) \propto M^{\nu}/(\ln(M))^{\beta}$

Case (3) has attracted special interest [10, 11]. For an exponent $\beta = \frac{1}{2}$ case (3) implies a multiscaling behaviour of the mass density $M(x) \propto r^{D(x)-1}$ where $D(x)$ is a non-constant function of $x = r/R_G$.

In the following we will show that case (1) combined with the assumption of a Gaussian distribution leads to an unphysical singularity in the mass distribution in the limit of infinite M. This result is a consequence of the fact that the Gaussian does not drop fast enough at the centre of the cluster, which suggests that $\nu' = \nu$ cannot be true.

On the other hand, we find numerically that a Gaussian is not a good description for the growth probabilities near the seed of the cluster, since for small r the tail behaves like a power law [12]. Furthermore, one has to realize that $P(r, M)$ drops to zero for large r simply because of the finite size of the cluster. Usually such an effect would be taken into account by considering a finite size cutoff function.

Here, we study the growth probabilities $P(r; M)$. We show that the numerical data is only consistent with a Gaussian behaviour around the maximum but that this description indeed fails in the small r tail. We show that another type of behaviour $P(r, M) \propto (r^{\alpha}/\xi) g((r - r_0)/\xi)$—where $g(x)$ is a scaling function—is equally consistent with our data around the maximum as in the tail. When this power law term is included in the distribution the previously mentioned singularity disappears.

Thus, the organization of the paper is as follows. In section 2 we analyse the consequences of the different possible behaviours of $\xi(M)$ on the mass density. In section 3 we study numerically the behaviour of the growth probabilities and in section 4 we show the consequences of the power law tail on the mass density.

2. Analytical results

Following [7], we define two lengths, the radius $r_0(M)$ and the width $\xi(M)$, as $r_0(M) \equiv \langle r \rangle$, $\xi^2(M) \equiv \langle r^2 \rangle - \langle r \rangle^2$, and $r = |\mathbf{r}|$. Here, $\langle f(r) \rangle = \int f(r) P(r, M)\, dr$. Let us assume that $P(r, M)$ can be described by a Gaussian distribution

$$P(r, M) = \frac{1}{[2\pi\xi^2(M)]^{1/2}} \exp[-(r - r_0(M))^2/2\xi^2(M)]. \tag{3}$$

Here, we check whether the above form of $P(r, M)$ is consistent with the growth of a fractal. We start from the relation

$$N(r, M) = \int_1^M P(r, M')\, dM' \tag{4}$$

where $N(r, M)\, dr$ is the number of sites within $[r, r + dr]$ in clusters of mass $< M$. Equation (4) can be derived from the fact that any site that lies within $[r, r + dr]$, in a cluster of mass M, has to be attached at the previous stage of growth $(1 < M' < M)$ [9].

Case (1): Let us assume that $r_0(M) \equiv AM^\nu$ and $\xi(M) \equiv BM^\nu$, where A, B are constants and we neglect other corrections. Using (4), we get

$$N(r,M) = \frac{1}{[2\pi B^2]^{1/2}} \int_1^M \frac{1}{M'^\nu} \exp[-(r - AM'^\nu)^2/(2B^2M'^{2\nu})]\,\mathrm{d}M'$$
$$= \frac{1}{[2\pi B^2]^{1/2}} \exp\left(-\frac{A^2}{2B^2}\right) \int_1^M \frac{1}{M'^\nu} \exp\left(-\frac{r^2}{2B^2M'^{2\nu}} + \frac{rA}{B^2M'^\nu}\right) \mathrm{d}M'. \qquad (5)$$

Changing the variable of integration to $x \equiv M'^{-\nu}$, equation (5) becomes

$$N(r,M) = \frac{1}{\nu[2\pi B^2]^{1/2}} \exp\left(-\frac{A^2}{2B^2}\right) \int_{M^{-\nu}}^1 \frac{1}{x^{1/\nu}} \exp\left(-\frac{r^2x^2}{2B^2} + \frac{rAx}{B^2}\right) \mathrm{d}x. \qquad (6)$$

The total mass contained in the $[r, r+\mathrm{d}r]$ shell is, by definition, $N(r,M)\,\mathrm{d}r$. For a fixed value of r, $N(r,M)$ becomes larger for larger values of M. Especially, in the $M \to \infty$ limit, the integral diverges due to the singularity at $x=0$ since $1/\nu$ is larger than unity. This divergence is inconsistent with the well-established fractal description of DLA.

Case (2): Consider the case $\xi \equiv BM^{y\nu}$ with $0<y<1$. The integral that corresponds to (5) becomes

$$N(r,M) = \frac{1}{[2\pi B^2]^{1/2}} \int_1^M \frac{1}{M'^{y\nu}} \exp[-(r - AM'^\nu)^2/(2B^2M'^{2y\nu})]\,\mathrm{d}M'. \qquad (7)$$

Changing the integration variable to $x \equiv M'^{-\nu}$, we get

$$N(r,M) = \frac{1}{\nu[2\pi B^2]^{1/2}} \int_{M^{-\nu}}^1 x^{y-1-1/\nu} \exp\left(-\frac{A^2}{2B^2}x^{2(y-1)} - \frac{r^2}{2B^2}x^{2y} + \frac{rA}{B^2}x^{2y-1}\right) \mathrm{d}x. \qquad (8)$$

In the $M \to \infty$ limit equation (8) becomes

$$N(r,M) = \frac{1}{\nu[2\pi B^2]^{1/2}} \int_0^1 x^{y-1-1/\nu} \exp\left[-\frac{x^{2y-2}}{2B^2}(rx-A)^2\right] \mathrm{d}x$$
$$\propto r^{1/\nu - 1}. \qquad (9)$$

Since $D_f \equiv 1/\nu$, equation (9) becomes $N(r,M) \sim r^{D_f - 1}$, consistent with the fact that the cluster is fractal with dimension D_f.

Case (3): Finally, consider the case $\xi \equiv BM^\nu/(\ln M)^{1/2}$.

$$N(r,M) = \frac{1}{[2\pi B^2]^{1/2}} \int_1^M \frac{(\ln M')^{1/2}}{M'^\nu} M'^{-(1/2B^2)(rM'^{-\nu} - A)^2}\,\mathrm{d}M'. \qquad (10)$$

Changing the variable to $x \equiv M'^{-\nu}$, equation (10) becomes

$$N(r,M) = \frac{1}{[2\pi\nu^3B^2]^{1/2}} \int_{M^{-\nu}}^1 (-\ln x)^{1/2} x^{-1/\nu + (rx-A)^2/(2\nu B^2)}\,\mathrm{d}x. \qquad (11)$$

Since the logarithmic correction to the width appears not only in the normalization of the Gaussian but also in its exponent, the behaviour of the integrand is now substantially changed as compared to case (1). There, one observes a behaviour

$x^{-1/\nu}\exp(-ax^2+bx+c)$ (equation (6)), whilst here one obtains $x^{-1/\nu+ax^2+bx+c}$. The integral in (11) cannot be evaluated in a closed form, but we can still extract some of the properties. First, consider the $M\to\infty$ limit. The logarithmic term in the integrand is divergent at $x=0$ if the exponent of the power law term $(A^2-2B^2)/(2\nu B^2)$ is smaller than -1. Therefore the whole integral is convergent only if $A^2>2(1-\nu)B^2$. If the integral in (11) is convergent, the convergent value can be estimated by treating the $\ln x$ term as a constant. The resulting form is consistent with the idea of multiscaling.

The analysis presented above shows that if $P(r,M)$ is Gaussian, the width of $P(r,M)$, $\xi(M)$, cannot scale as R_G. The divergence in case (1) suggests that there is not enough 'screening' at the inside of the cluster. Therefore, $\xi(M)$ should increase more slowly than R_G as M is increased.

In the next section, we numerically check whether the $P(r,M)$ can be well described by a Gaussian.

3. Numerical simulations

To measure the growth probabilities $P(x,M)$ numerically (here we scale $x=r/R_G$) we generate clusters of masses 10 000, 20 000, 50 000, 100 000, 200 000 and 500 000 particles. For each mass we generate at least 150 different clusters. Only for the 100 000 particle clusters we grow 1000 samples to have very good statistics and to see the details of the distribution. The growth probabilities are obtained in a static measurement: first we grow each cluster to the full size and afterwards we probe its surface using another 10^6 rendom walkers. Using a fast algorithm [11], the generation of one 100 000 particle cluster and the measurement of $P(x,M)$ can be done within 90 minutes on a Sun SPARC station 2. To obtain the $P(x,M)$ we bin it for $x\in[0,2.5]$ at 64 equidistant points and count the number of particles touching the cluster at a radius between x and $x+\mathrm{d}x$. Figure 1 shows the growth probabilities for all cluster masses in linear and logarithmic scale. One has to notice the power law tail of $P(x,M)$ at small x, which is a property that cannot be found for pure Gaussians. To smooth out the noise we show in figure 2 the integrated distributions $IP(x,M)=\int_0^x P(x',M)\,\mathrm{d}x'$. For small cluster masses—up to 100 000 particles—one determines a systematic decrease of the width. For the largest cluster masses no such clear systematic behaviour can be

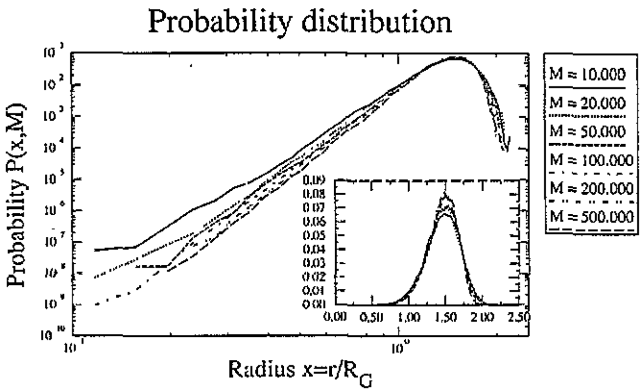

Figure 1. Growth probability distributions for different cluster masses in linear and double logarithmic scale.

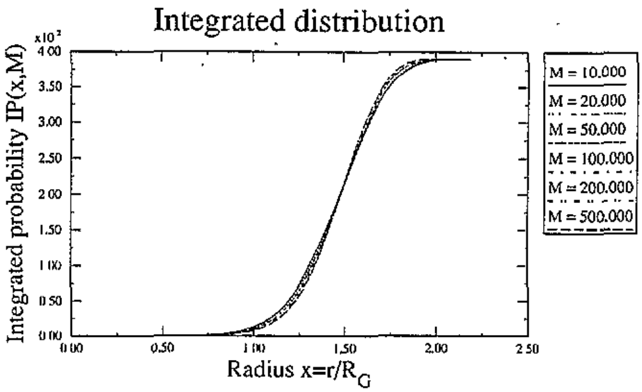

Figure 2. Integrated probability distributions.

seen. The curves for the 200 000 and 500 000 particle clusters lie on top of each other. The width ξ can be studied quantitatively with two methods. The first one determines the width by measuring the first and second moments of the data $\xi/R_G = [\langle x^2\rangle - \langle x\rangle^2]^{1/2}$ and the second consists in fitting a Gaussian to the data using a nonlinear fitting routine. Both methods give comparable results and one obtains

$$\begin{aligned} \xi_{\text{Moments}} &\propto R_G M^{-0.044\pm0.003} \\ \xi_{\text{Gauss}} &\propto R_G M^{-0.050\pm0.003}. \end{aligned} \tag{12}$$

We also check whether our data is consistent with case (3) and we find the behaviour

$$\xi_{\text{Moments}} \propto \frac{R_G}{(\ln(1.07M))^{1/2}}. \tag{13}$$

In figure 3 we show the data of the width ξ_{Moments}/R_G as a function of the cluster mass in semi-logarithmic scale. The solid lines in this plot are the results obtained by fitting a power law and a $1/(\ln M)^{1/2}$ law to our data. For cluster masses up to 200 000 particles one seems to have good agreement with the assumption of a logarithmic correction.

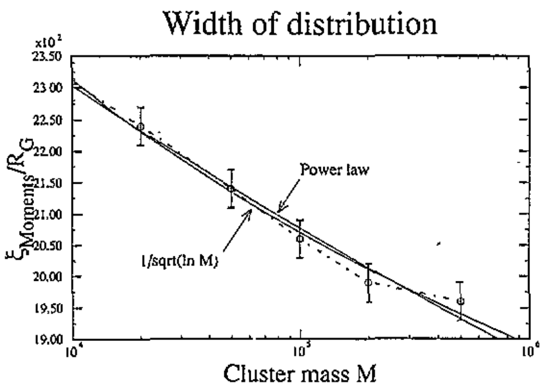

Figure 3. Scaling of the width of the distribution as obtained by measuring the moments of the distribution. The two solid lines denote the fits of a power law and a $1/(\ln M)^{1/2}$ law to our data.

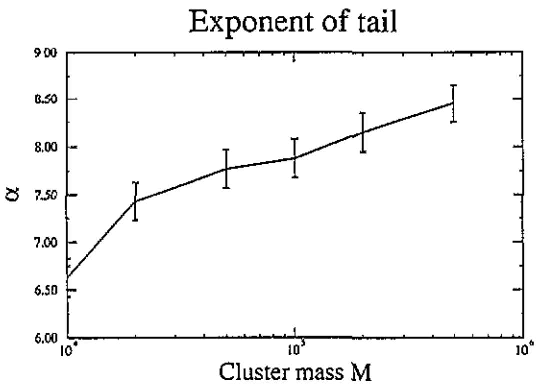

Figure 4. Behaviour of the 'apparent' exponents of the power law with varying cluster mass. This demonstrates, that the mass dependence is important in order to obtain the correct exponent.

Thus, our data seems to be inconsistent with case (1) $\nu' = \nu$. For clusters up to 200 000 particles our data are consistent with cases (2) and (3).

Here, we want to study in more detail the form of the inner part of $P(x, M)$. Since figure 1 shows a clear power law behaviour for small x, we assume a simple scaling law

$$P(x, M) \propto \frac{x^{\alpha}}{\xi/R_G} f\left(\frac{x - x_0}{\xi/R_G}\right) \tag{14}$$

where $f(x)$ is some scaling function. This type of behaviour is used to stress the power law behaviour of the small x tail.

If one uses for $f(x)$ too simple a function—like a Fermi function—which becomes constant for small x, one does not describe the full mass dependence of $P(x, M)$ correctly: one measures only apparent large exponents α, which increase logarithmically with M (figure 4).

To take this mass dependence correctly into account we perform a data collapse to the scaling form (14). Here we use for the width ξ/R_G and centre x_0 the values calculated from the moments of $P(x, M)$ and vary the exponent α. The best collapse of our data, which is shown in figure 5, we obtain for a value $\alpha = 2.0 \pm 0.4$. This value of α is much smaller than those obtained previously [2]. Thus, the previously measured large exponents are only 'apparent' values. They are combinations of the power law tail and the mass dependence of the width of the scaling function. If one takes into account the mass dependence of the scaling function, α decreases significantly. However, we still find a strong power law screening of the inner parts of the cluster.

4. Consequences

Having established the existence of the power law tail of $P(r, M)$, we study how this modifies the results in section 2. Here we assume that $P(r, M)$ can be described by equation (14) with a normalization factor N_0. Let the average of $f(x)$ be zero and the

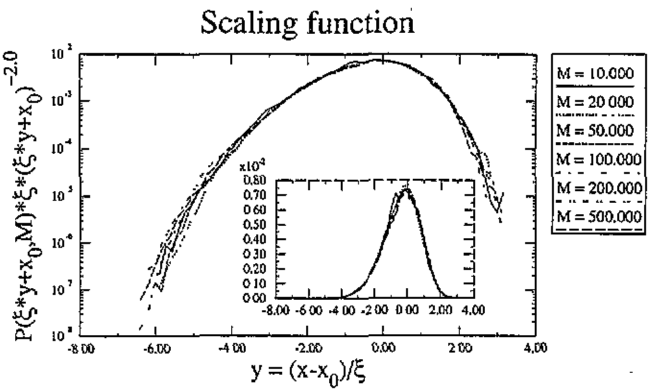

Figure 5. Plot of the scaling function $g(y) \propto P(\xi y + x_0, M)\xi(\xi y + x_0)^{\alpha}$. The maximum of the distribution collapses as well as the power law tail.

width be some constant. We first consider case (1). Since $P(r, M)$ has to be normalized, it is

$$\int_0^{\infty} \frac{N_0}{\xi}\left(\frac{r}{R_G}\right)^{\alpha} f\left(\frac{r-r_0}{\xi}\right) \mathrm{d}r = 1. \tag{15}$$

Changing the integration variable to $x = r/\xi$ equation (15) becomes

$$N_0\left(\frac{\xi}{R_G}\right)^{\alpha} \int_0^{\infty} x^{\alpha} f(x - x_0)\, \mathrm{d}x = 1 \tag{16}$$

where $x_0 \equiv r_0/\xi$. Since the integral is a constant, N_0 should scale as $(R_G/\xi)^{\alpha}$. The equation that corresponds to (5) is

$$\begin{aligned} N(r, M) &\propto \int_1^M \frac{1}{\xi}\left(\frac{r}{\xi}\right)^{\alpha} f\left(\frac{r-r_0}{\xi}\right) \mathrm{d}M \\ &\propto \int_1^M M^{-(1+\alpha)/D_f} r^{\alpha} f\left(\frac{r-r_0}{\xi}\right) \mathrm{d}M. \end{aligned} \tag{17}$$

The part that causes a divergence in (5) is the $M^{-(1+\alpha)/D_f}$ term, since other terms in the integrand become constants for a fixed value of r. Therefore, the integral (17) diverges if the exponent $(1+\alpha)/D_f$ is less than unity, which is equivalent to $\alpha < D_f - 1$. Since the numerical value of α is obviously larger than $\tilde{\alpha} = D_f - 1 \simeq 0.71$, our Ansatz is no longer excluded by the self-consistency argument. One can show, in a similar way, that the existence of a power law tail is consistent with the other cases (2) and (3).

5. Conclusion

We have studied the growth probabilities for large off-lattice DLA clusters. We have shown that the usual assumption of a Gaussian behaviour leads to unphysical singularities under the assumption that the width of the Gaussian scales with the same exponent as its centre. We have shown that extensive numerical calculations seem to be inconsistent with the assumption $\nu' = \nu$. It seems that cases (2) or (3) have to be favoured. Moreover, we showed that $P(r, M)$ has a power law tail which is inconsistent with the Gaussain behaviour that is usually assumed. Therefore we suggested another type of behaviour $P(r, M) \propto (r^{\alpha}/\xi)f((r-r_0/\xi))$ which removes the unphysical singularity described above. This type of behaviour describes the maximum of the measured growth probabilities as well as the tail. We showed that one explicitly has to take into account the dependence of the width of the distribution on the cluster mass in order to find the correct exponent $\alpha \approx 2$. If one uses a simple *ad hoc* assumption to determine the inner tail of the distribution one mixes the 'real' power law term and contributions from the scaling function. Unfortunately, our data does not enable us to draw any conclusions about the behaviour of the minimum growth probabilities, as studied for example in [5]. The data we measured is only an average of the growth probabilities within one 'shell' of the cluster. Thus, we only have data about the spatial distribution of growth probabilities and not about the distribution $n(P)$ of the growth probabilities themselves. A measurement of $n(P)$ is, for small P, probably not feasible with a random walker method.

Acknowledgments

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